Numerical Optimal Control of the Wave Equation: Optimal Boundary Control of a String to Rest in Finite Time

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Abstract. In many real-life applications of optimal control problems with constraints in form of partial differential equations (PDEs), hyperbolic equations are involved which typically describe transport processes. Since hyperbolic equations usually propagate discontinuities of initial or boundary conditions into the domain on which the solution lives or can develop discontinuities even in the presence of smooth data, problems of this type constitute a severe challenge for both theory and numerics of PDE constrained optimization.

In the present paper, optimal control problems for the well-known wave equation are investigated. The intention is to study the order of the numerical approximations for both the optimal state and the optimal control variables for problems with known analytical solutions. The numerical method chosen here is a full discretization method based on appropriate finite differences by which the PDE constrained optimal control problem is transformed into a nonlinear programming problem (NLP). Hence we follow here the approach ‘first discretize, then optimize’, which allows us to make use not only of powerful methods for the solution of NLPs, but also to compute sensitivity differentials, a necessary tool for real-time control.

Keywords. Optimal control of partial differential equations; optimal control of hyperbolic equations; optimal control of the wave equation; first discretize, then optimize.

1. Introduction

Owing to its importance for engineering applications, the field of PDE constrained optimization has become increasingly popular. Mathematicians, computer scientists, and computational engineers are confronted with very complex optimization problems which presently allow only the application of simulation-based methods. Therefore, there is a strong need for new efficient optimization methods which are capable of tackling real-life engineering applications, including real-time applications. Those problems may be nonlinear and consist of coupled systems of high dimension and different type with complicated right hand sides and possibly non-standard initial and boundary conditions; see, e.g., the highly complex nonlinear PDE system describing the dynamical behaviour of certain fuel cell systems [5], [6], and [24].

Without doubt Lions’s book [23] is still the standard for optimal control problems with linear equations and convex functionals. Nonconvex problems with semilinear equations of elliptic and parabolic type are in the focus of Tröltzsch’s new book [26], particularly addressing questions of existence of solutions and optimal controls, the derivation of necessary conditions and adjoint equations as well as of second order sufficient conditions.

While the theory of optimal control for elliptic and parabolic equations is well developed in the semilinear case, although the associated optimal control problems generally are nonconvex, hyperbolic equations are not as well understood despite the fact that many dynamical processes are, at least partly, of hyperbolic nature; see, e.g., the aforementioned references on fuel cell control.

Linear hyperbolic equations are treated, besides in Lions’s book, in Ahmed and Teo [1]. An introduction into the control of vibrations can be found in Krabs [18]. Oscillating elastic networks are investigated by Lagnese, Leugering et al. [19]–[20], [22], Gugat [10], and in the new book by Dáger and Zuazua [7]; see also the references cited therein. However, semilinear hyperbolic problems are still not well understood because of the weaker smoothness of the solution operators. The most recent progress on (optimal) control of hyperbolic equations is due to S. Ulbrich [27], [28] and Zuazua [29], [30].

In this paper we will investigate a well understood optimal control problem concerned with the classical wave equation. The present paper is particularly based on a series of papers by Gugat [9], [11], [12], [14] and Gugat et. al. [15], [16]. Therein analytical solutions for optimal boundary control problems are
presented with the control costs measured in the $L^p$-norm for $1 \leq p \leq \infty$ as performance index to be minimized and the wave equation as the major constraint. In contrast, we are going to solve these problems numerically in order to show the order of approximation for their optimal solutions. For, we want to promote an approach for the numerical solution of PDE constrained optimal control problems which also works if hyperbolic equations are involved. The method of choice proposed here is either a full discretization method, in case of small size problems, or the vertical method of lines, in case of medium size problems. For more references on this particular problem, both from theoretical and numerical points of view, it is referred to [16].

For large size problems only model reduction methods may today give a chance for their solution such as proper orthogonal decomposition; see, e.g., Hinze, Volkwein [17]. Concerning small and medium size problems, appropriate difference methods for the spatial discretization, resp. semidiscretization must be applied in order to approximate the hyperbolic equation correctly. Clearly, both approaches belong to the class ‘first discretize, then optimize’. They possess the advantage that the full power of methods for nonlinear programming problems (NLPs) and ODE (ordinary differential equation) constrained optimal control problems can be used including their possibilities for a numerical sensitivity analysis and therefore real-time control purposes; see, e.g., Büskens [2]–[4]. The latter is a must if optimal solutions are to be applied in practise. No doubt, that the approach ‘first optimize, then discretize’ generally may give more theoretical inside and safety, but as in ODE constrained optimization its advantage does not pay for its drawbacks when complicated real-life applications are to be treated.

2. Optimal Control Problems

2.1 The infinite-dimensional optimal control problem

Let the length $L > 0$, the time $T > 0$, and the wave velocity $c > 0$ be given real numbers as well as $p \in [1, \infty)$.

Defining the norm

$$\|(v, w)\|_{p, [0, T]} := \left( \int_0^T (|v(t)|^p + |w(t)|^p) \, dt \right)^{\frac{1}{p}},$$

one can state the following optimal control problem:

$$\|(y_L, y_R)\|_{p, [0, T]}^p \triangleq \min$$

subject to the constraints

$$y_L, y_R \in L^p[0, T],$$

$$u_t(x, t) = c^2 u_{xx}(x, t), \quad (x, t) \in (0, L) \times (0, T),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, L],$$

$$u(0, t) = y_L(t), \quad u(L, t) = y_R(t), \quad t \in [0, T],$$

$$u(x, T) = 0, \quad x \in [0, L], \quad \text{and} \quad u_t(x, T) = 0, \quad x \in (0, L).$$

Hence we have to control a vibrating string with the state variable $u(x, t)$, satisfying the wave equation (3), from an arbitrary initial state (4) to rest at a prescribed final time (6) by means of the boundary conditions (5) via the control functions (2) such that the functional (1) is minimized.

The question of exact controllability for this type of problem has been answered in [12] (for the case $p = 1$) and in [16] (for the case $p \in [2, \infty)$). Under the assumption that the final time satisfies $T \geq L/c$, i.e., that the control time has to be greater or equal the time the wave needs to travel from one end of the string to the other (the characteristic time), the initial-boundary value problem (3)–(5) has a weak solution satisfying the terminal conditions (6), if and only if the function $u_0$ be in the Hilbert space $L^p[0, L]$ and the function $u_1$ be such that the mapping $x \rightarrow \int_0^x u_1(s) \, ds$ is also in $L^p[0, L]$, i.e., $u_1$ is in the Sobolev space $W^{-1, p}(0, L)$. In [16] also the case $p = \infty$ as well as the related problem of steering the system from zero state to a prescribed terminal state has been investigated. Then the terminal state also has to belong to these function spaces.
Moreover, in [11] and [12], Theorem 2, analytical solutions for the optimal controls for the case \( p = 1 \) are given which are based on the wellknown D’Alembert’s solution of the wave equation. Note, the solution to (1)–(6) is not unique, if \( p = 1 \). For \( p = \infty \) the solution is also not unique (see [9]), but it is unique for \( p \in (1, \infty) \) (see [16]). Whereas for \( p = 1 \) all solutions are known (see [12]), for \( p = \infty \) only the element of the solution set is known having minimal \( L^2 \)-norm (see [9]).

In this paper we deal only with the case \( p = 2 \), so the functional (1) does not lack smoothness as it would do in case of \( p = 1 \). This choice avoids artificial difficulties for our numerical tests. Moreover, we know that the optimal solution of the \( L^2 \)-version is unique and can be found among the optimal solutions for \( p = 1 \); see [16]. This makes the \( L^2 \)-version particularly valuable for numerical tests. Of course, the optimal values of the functionals vary with the parameter \( p \).

Henceforth, we minimize

\[
\|(y_L, y_R)\|_{L_2([0,T])}^2 := \int_0^T (|y_L(t)|^2 + |y_R(t)|^2) \, dt \leq \min.
\]

### 2.2 The finite-dimensional optimal control problem

Introducing the time step size \( k > 0 \) and the spatial step size \( h > 0 \) so that \( q := T/k \) and \( m := L/h \) are natural numbers, the discrete version of (7) subject to (2)–(6) is given by the following quadratic programming problem (QP)

\[
k \cdot \sum_{j=1}^{q-1} (|y_L(t_j)|^2 + |y_R(t_j)|^2) \leq \min
\]

subject to the constraints

\[
Q_{h,k} u_i^j(Y_{L,k}, Y_{R,k}) = 0,
\]

\[
u(x_i,0) = u_0(x_i), \quad u_i(x_i,0) = u_1(x_i), \quad i = 0, \ldots, m, \tag{9}
\]

\[
u(x_0, t_j) = y_L^j, \quad u(x_m, t_j) = y_R^j, \quad j = 0, \ldots, q, \tag{10}
\]

\[
u(x_i, t_q) = 0, \quad i = 0, \ldots, m, \quad \text{and} \quad \frac{u(x_i, t_q) - u(x_i, t_{q-1})}{k} = 0, \quad i = 1, \ldots, m - 1. \tag{11}
\]

where \( x_i := i h, \quad t_j := j k, \quad u_i^j := u(x_i, t_j), \quad y_L^j := y_L(t_j), \quad y_R^j := y_R(t_j), \quad Y_{L,k} := (y_L^0, \ldots, y_L^q)^\top, \) and \( Y_{R,k} := (y_R^0, \ldots, y_R^q)^\top \). The difference operator \( Q_{h,k} \) represents an appropriate discretization scheme for the wave equation specified later. In the trapezoidal rule formula for the objective function, the first and last term which have to be multiplied by a factor \( \frac{1}{2} \) are dropped because the associated control values are prescribed by the initial and terminal conditions. Note that consistency conditions, i.e. \( u_0(0) = y_L(0), \) \( u_0(L) = y_R(0), \) \( u(0,T) = y_L(T) = 0, \) and \( u(L,T) = y_R(T) = 0 \) are automatically satisfied here, although they cannot be required in the infinite-dimensional problem because they constitute pointwise evaluations.

### 3. Numerical Results

Numerical tests are performed for the following four examples:

**Example 1:** Let be \( L = 1, \quad T = 3.25, \quad c = 1, \) \( u_0(x) = 0, \) \( u_1(x) = \sin(\pi x). \)

**Example 2:** Let be \( L = \pi, \quad T = 2 \pi, \quad c = 1, \) \( u_0(x) = \sin x, \) \( u_1(x) = \cos x. \)

**Example 3:** Let be \( L = \pi, \quad T = 2 \pi, \quad c = 2, \) and

\[
u_0(x) = \begin{cases} \frac{-1}{4} x & \text{for } 0 \leq x \leq \frac{1}{4} \pi, \\ \frac{1}{2} x - 2 & \text{for } \frac{1}{4} \pi \leq x \leq \frac{3}{4} \pi, \quad \text{and} \quad u_1(x) = 0, \quad x \in [0, \pi]. \\ \frac{-1}{4} x + 4 & \text{for } \frac{3}{4} \pi \leq x \leq \frac{5}{4} \pi, \end{cases}
\]
Example 4: Let be $L = 2$, $T = 6$, $c = 1$, and
\[ u_0(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ x - 2 & \text{for } 1 \leq x \leq 2, \end{cases} \quad \text{and } u_1(x) = 0, \quad x \in [0, 2]. \]

All computations were carried through by means of MATLAB. Despite the fact that the problem (8)-(12) is a QP, the MATLAB-routine `fmincon`, an SQP (sequential quadratic programming) method, has been used here, in order to make the method applicable also to (more) nonlinear problems.

When applying numerical difference schemes to the wave equation we have two possibilities. The first class of difference schemes is based on an equivalent formulation of the Cauchy problem associated with the wave equation, i.e.,
\[ u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (13) \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, L], \quad (14) \]
as follows,
\[ u_t(x, t) + c v_x(x, t) = 0, \quad (15) \]
\[ v_t(x, t) + c u_x(x, t) = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (16) \]
\[ u(x, 0) = u_0(x), \quad (17) \]
\[ v(x, 0) = -\frac{1}{c} \int_0^x u_1(s) \, ds + C, \quad x \in [0, L]. \quad (18) \]
The constant $C$ is arbitrary and can be set to zero since the component $v$ is determined up to a constant only.

The second class tackles the wave equation (13) directly.

The following well-known schemes are candidates for an application to the system (15), (16), if certain restrictions are obeyed, see, e.g., Strikwerda [25]:

The FTFS scheme (‘forward-time forward-space’) being an upwind scheme for $c < 0$, explained below:
\[ \frac{u_i^{j+1} - u_i^j}{k} + c \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h} = 0; \quad (19) \]
the FTBS scheme (‘forward-time backward-space’) being an upwind scheme for $c > 0$, explained below:
\[ \frac{u_i^{j+1} - u_i^j}{k} + c \frac{u_i^j - u_{i-1}^{j+1}}{h} = 0; \quad (20) \]
the Lax-Friedrich scheme:
\[ \frac{u_i^{j+1} - \frac{1}{2} (u_{i+1}^{j+1} + u_{i-1}^{j+1})}{k} + c \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} = 0; \quad (21) \]
the Lax-Wendroff scheme:
\[ u_i^{j+1} = u_i^j - \frac{c k}{2h} \left( u_{i+1}^j - u_{i-1}^j \right) + \frac{c^2 k^2}{2h^2} \left( u_{i+1}^j - 2 u_i^j + u_{i-1}^j \right). \quad (22) \]
In contrast, the following scheme directly applies to the wave equation (13). Approximating both second derivatives by central difference quotients of second order one obtains
\[ u_i^{j+1} = 2 \left( 1 - \alpha^2 \right) u_i^j + \alpha^2 \left( u_{i+1}^j + u_{i-1}^j \right) - u_i^{j-1} \quad \text{for } j \geq 1, \quad (23) \]
\[ u_i^1 = (1 - \alpha^2) u_i^0 + \frac{1}{2} \alpha^2 \left( u_{i+1}^0 + u_{i-1}^0 \right) + k u_1(x_i) \quad (24) \]
with $\alpha := c \frac{k}{h}$. Equation (24) serves as the second starting line on $t = k$ for the multistep method (23) besides the initial conditions (14) on the line $t = 0$. 

All of the above schemes can be rewritten so that $u_{i}^{j+1}$ is a linear combination of values of $u$ at time instances $j$ and $j - 1$. For example, the formula

$$u_{i}^{j+1} = (1 + c p) u_{i}^{j} - c p u_{i}^{j+1} =: P_{h,k} u_{i}^{j}$$

with $p := \frac{k}{h}$

represents the FTFS scheme. Hereby $P_{h,k}$ denotes the difference operator for the FTFS scheme, resp. we define $Q_{h,k} u_{i}^{j} := u_{i}^{j+1} - P_{h,k} u_{i}^{j} = 0$. Analogously we can rewrite all schemes except (23), (24) as one step method, i.e.,

$$u_{i}^{j+1} = P_{h,k} u_{i}^{j} .$$

In order to be convergent, these schemes must be consistent and stable according to the well-known Lax Equivalence Theorem. More precisely, a consistent finite difference scheme for a PDE, for which the Cauchy problem is wellposed, is convergent (e.g. w.r.t. the $L^2$-norm) if and only if it is stable; see for example [25]. Note that this result does not apply to the optimal control functions; see [29], [30].

In order to prove that, it is sufficient to show convergence solely for the simple scalar transport equation

$$u_t(x,t) + c u_x(x,t) = 0 ,$$

since the system (15), (16) can be decoupled by a similarity transformation yielding a hyperbolic system with eigenvalues $\lambda_{1,2} = \pm c$; see, e.g., Larsson, Thomée [21].

It is then an easy exercise to show that the upwind schemes are both of consistency order $O(k) + O(h) =: (1,1)$. However, the FTFS scheme is stable only if $c < 0$ and $-1 \leq cp \leq 0$, whereas the FTBS scheme is stable only if $c > 0$ and $0 \leq cp \leq 1$. This explains the names: The spatial difference quotients have to be chosen 'against the wind', i.e., against the direction of the information flow.

The situation for the Lax-Friedrich scheme is more complicated. It is only consistent of order $(1,1)$ if $k^{-1} h^2 \rightarrow 0$, which can be fulfilled if $k = \Lambda(h)$ with a sufficiently smooth function $\Lambda$ with $\Lambda(0) = 0$. The scheme is stable if $|cp| \leq 1$, and hence convergent then.

The Lax-Wendroff scheme is consistent of order $O(k^2) + O(h^2) =: (2,2)$ and stable if also $|cp| \leq 1$, hence convergent then.

In case of general hyperbolic systems, the CFL condition $|cp| \leq 1$ has to be replaced by $|\lambda p| \leq 1$ for all eigenvalues $\lambda$.

### 3.1 Results for Example 1

The following two figures show the results for Example 1 obtained by means of the multistep scheme (23), (24). Exact and approximated optimal boundary controls coincide perfectly; see Fig. 1 (left). The associated state is also given in Fig. 1 (right).

Figure 1: Example 1.

**Left:** Exact and approximate left and right optimal boundary controls $y_L^{\text{exact}}(t)$ and $y_R^{\text{exact}}(t)$, resp. $y_L^{\text{app}}(t)$ and $y_R^{\text{app}}(t)$ using the multistep scheme (23), (24) with $h = k = 0.000625$.

**Right:** Approximate optimal state variable $u(x,t)$ on the time interval $[0,3.25]$ using the multistep scheme (23), (24) with $h = k = 0.0125$.
3.2 Results for Example 2

Figure 2 shows the results for Example 2 obtained by means of the Lax-Wendroff scheme (22). Again the coincidence between the numerical approximation and the exact optimal solution is convincing. Solely at $t = \pi$, where the first derivative of the optimal boundary control is discontinuous, a slight oscillation arises.

![Figure 2: Example 2.](image)

**Left:** Exact and approximate left and right optimal boundary controls $y_L^{\text{exact}}(t)$ and $y_R^{\text{exact}}(t)$, resp. $y_L^{\text{app}}(t)$ and $y_R^{\text{app}}(t)$.

**Right:** Approximate optimal state variable $u(x, t)$ on the time interval $[0, 2\pi]$, both computed with the Lax-Wendroff scheme (22) using $h = k = \pi/140$.

3.3 Results for Example 3

Figure 3 shows the results for Example 3 obtained by means of the Lax-Friedrich scheme (21). Again the coincidence between the numerical approximation and the exact optimal solution is perfect. However, note that these results can only obtained if $c p = 1$ is chosen which is the maximum value of the stability interval. The same result holds when using the multistep scheme (23), (24). For $c p < 1$ oscillations occur for both methods.

![Figure 3: Example 3.](image)

**Left:** Exact and approximate left and right optimal boundary controls $y_L^{\text{exact}}(t)$ and $y_R^{\text{exact}}(t)$, resp. $y_L^{\text{app}}(t)$ and $y_R^{\text{app}}(t)$.

**Right:** Approximate optimal state variable $u(x, t)$ on the time interval $[0, \pi]$, both computed with the Lax-Friedrich scheme (21) using $h = \pi/101$ and $k = \pi/202$. 
3.4 Results for Example 4

For Example 4, Fig. 4 shows the superiority of the second order multistep scheme (23), (24) compared to the simple first order upwind scheme (20) w.r.t. the resolution of the discontinuities despite the larger step size of the multistep method. The associated approximate state variables are given in Fig. 5.

Figure 4: Example 4.
Exact and approximate left and right optimal boundary controls $y_{L}^{\text{exact}}(t)$ and $y_{R}^{\text{exact}}(t)$, resp. $y_{L}^{\text{app}}(t)$ and $y_{R}^{\text{app}}(t)$,
left: using the upwind scheme (20) with $h = k = 0.01$,
right: using the multistep scheme (23), (24) with $h = k = 0.05$

Figure 5: Example 4.
Approximate optimal state variable $u(x, t)$ on the time interval $[0, 6]$,
left: computed with the upwind scheme (20),
right: computed with the multistep scheme (23), (24), both using $h = k = 0.05$

3.5 Numerical Convergence Analysis

For the weighted step size ratio $c_p = 1$ we were able to produce very precise results for all optimal control examples investigated. The following Tables 1 and 2 list the discrete $L^2$-norm errors for the approximated optimal control variables and the numerically achieved convergence orders for all difference schemes investigated for Example 2. Similar results were obtained for Example 1 and 4; see [8]. Note that the results for the first order methods (20), (21), and (22) coincide up to the first four digits so that they are together given in Table 2. The results for Example 3 are given in Tables 3 and 4 below.

As we can see, the convergence order drops down to 0.5 for both the first and the second order methods which may be due to the nonsmooth control variables.
Table 1: Numerical convergence and error analysis of the multistep method (23), (24) for Example 2

<table>
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<tr>
<th>h</th>
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<th>N</th>
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<th>order $|.|_2$</th>
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<td>0.0098</td>
<td>321</td>
<td>0.0264</td>
<td>0.4983</td>
</tr>
</tbody>
</table>

Table 2: Numerical convergence and error analysis of the upwind scheme (20), the Lax-Friedrich scheme (21), and the Lax-Wendroff scheme (22) for Example 2

For Example 3 the optimal control variables are continuous and, for both the second and the first order methods, the numerically achieved convergence order (w. r. t. the control functions) seems to be 1.5, which may be surprising at the first glance particularly concerning the first order methods. However, note that the theoretical convergence orders (for the state function) obtained for smooth data are only a lower bound.

Table 3: Numerical convergence and error analysis of the multistep method (23), (24) for Example 3

<table>
<thead>
<tr>
<th>h</th>
<th>k</th>
<th>N</th>
<th>error $|.|_2$</th>
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Table 4: Numerical convergence and error analysis of the upwind scheme (20), the Lax-Friedrich scheme (21), and the Lax-Wendroff scheme (22) for Example 3

For more numerical results, e. g., for the optimal boundary control of a string to be steered to an arbitrary terminal state, see [8].
Conclusions

Optimal control problems for hyperbolic equations such as the classical wave equation are a particular challenge both for theory and numerics because of the existence of nonsmooth solutions. Numerical computations for the optimal boundary control of a string to rest in finite time, for which analytical solutions are known, have shown that the approach ‘first discretize, then optimize’ is able to produce reliable results if the step size ratio is chosen close to the stability limit. The numerically achieved convergence orders w. r. t. the control functions drops down to 0.5 for the first as well as the second order methods investigated in the paper, if the optimal boundary controls are discontinuous, whereas in the continuous case the order is 1.5 for both the first and second order methods. These results obviously give rise to further questions, for example what is the best possible convergence order in the case of nonsmooth optimal boundary controls, which usually appear in the optimal control of hyperbolic equations.

Final Remark: After the presentation of the results of this paper at the MathMOD 2006 in Vienna, Gugat presented some new results at the EURO 2006 in Reykjavik [13] concerning the exact convergence order for the multistep method (23), (24). Firstly, he could show that, by assuming k = h, the finite-dimensional problem has a non-unique solution if p = 1 with a convex solution set. Furthermore, the solution is unique if p > 1. These results are based on a discrete D’Alembert’s solution for the difference equation (23) showing that the optimal control of the finite-dimensional problem has exactly the same structure as the solution of the infinite-dimensional problem. It then follows that the discretization error is of order O(h) for piecewise Lipschitz initial data. Despite this partial result the question that has been asked at the end of the conclusion is still open.

Finally, in the paper [14] an optimal control version is investigated which has a continuous state. Hereby the costs of the control rates are minimized.

References


